Stochastic resonance in a pulse neural network with a propagational time delay

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Abstract

Stochastic resonance in a coupled FitzHugh-Nagumo equation with a propagational time delay is investigated. With an appropriate set of parameter values, *i.e.*, the frequency of the periodic input, the propagational time delay, and the coupling strength, a deterministic firing induced by additive noise is observed, and its dependence on the number of neurons is examined. It is also found that a network composed of two assemblies shows a competitive behavior under control of the noise intensity.

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1 Introduction

Recently, Stochastic Resonance (SR) has attracted considerable attention. This is a phenomenon where a weak input signal is enhanced by its background noise (for reviews, see Dykman et al., 1995; Gammaitoni et al., 1998; Wiesenfeld and Jaramillo, 1998).

Particularly, SR in neural systems has been widely investigated by numerous authors. Theoretically, SR in a single neuron model, *e.g.*, the leaky integrate-and-fire model (Bulsara et al., 1996; Shimokawa et al., 1999), or the FitzHugh-Nagumo model (Longtin, 1993; Wiesenfeld et al., 1994), has been often analyzed, and it has been proposed that the biological sensory systems utilize their background noise to improve their sensitivity to external inputs. Experimentally, it is observed that mechanoreceptor cells of a crayfish with additive noise (Douglass et al., 1993) and caudal photoreceptor interneurons of a crayfish with intrinsic noise (Pei et al., 1996) show the property of SR. In these works, the background noise is implicitly regarded as due to thermal fluctuations, but there may be more origins of fluctuation in neural systems, *e.g.*, internal chaos, or numerous synaptic inputs (Mato, (1998, 1999); Sakumura and Aihara, 1998). With such noise sources, SR may play more functional roles in neural systems, *e.g.*, the information processing in the brain. To treat such dynamics, SR in coupled systems must be considered, and some new features have been reported. Collins et al. (1995) investigated the dynamics of the ensemble of FitzHugh-Nagumo equations and showed that controlling the noise intensity is not required. Kanamaru et al. (1999) considered SR in a diffusively coupled FitzHugh-Nagumo equation, and investigated the dependence of the system on the coupling strength and the number of neurons.

In the present paper, we investigate a coupled FitzHugh-Nagumo equation with propagational time delay, which models the time for a pulse to propagate on the axon from the pre-synaptic neuron to the post-synaptic neuron. In Sec. 2, a coupled FitzHugh-Nagumo equation and the correlation coefficient which measures the correlation between two pulse trains are defined. In Sec. 3, the behavior of the system without the propagational time delay (Kanamaru et al., 1999) is briefly summarized. In Sec. 4, the system with a propagational time delay is considered. With an appropriate set of system parameters, a deterministic firing induced by additive noise is observed, and its dependence on the number of neurons is investigated. In Sec. 5, we construct a network composed of two assemblies and examine a competitive behavior in the network by controlling the noise intensity. Conclusions and discussions are given in the last section.

2 Coupled FitzHugh-Nagumo equation and correlation coefficient

In the present paper, we treat a coupled FitzHugh-Nagumo (FN) equation with a propagational time delay, written as

$$\tau \dot{u}_i = -v_i + u_i - u_i^3/3 + w g_i(t) + S(f;t) + \xi_i(t) \tag{1}$$

$$\dot{v}_i = u_i - b v_i + a, \tag{2}$$

$$g_i(t) = \begin{cases} \displaystyle\sum_{j \neq i} \frac{1}{N-1} \left(u_j(t-d_p) - u_i(t) \right) & \text{if } N \geq 2 \\ 0 & \text{if } N = 1 \end{cases}, \tag{3}$$

$$S(f;t) = \begin{cases} I & \text{if } t \leq h \bmod f^{-1} \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

$$\langle \xi_i(t) \xi_j(t') \rangle = D \delta_{ij} \delta(t-t'), \tag{5}$$

for $i, j = 1, 2, \cdots N$, where $\tau = 0.1$, $a = 0.7$, $b = 0.8$, u_i is the fast variable which denotes the internal state of the i-th neuron, v_i is the slow variable which represents the refractory period, w is the coupling strength, $g_i(t)$ is the coupling term, $S(f;t)$ is the periodic pulse train with height I, width h, and frequency f, $\xi_i(t)$ is Gaussian white noise with intensity D, δ_{ij} is Kronecker's delta, and d_p is the propagational time delay from the j-th neuron to the i-th neuron. Note that the noises for different neurons are statistically independent, the coupling strengths and the propagational time delays are uniform in the network, and the coupling is diffusive, *i.e.*, for a large enough w and $d_p = 0$ the neurons synchronize each other. With the above configurations, a single FN model shows a characteristic of an excitable system, namely, it has a stable rest state, and with an appropriate amount of disturbance it generates a pulse with a characteristic magnitude of height and width. The parameters of the periodic pulse train $S(f;t)$ are set as $f = 0.1$, $I = 0.15$, and $h = 0.3$. Note that the height I is so small that no neuron generates a pulse without noise.

By the symmetry of the system, the behaviors of all the neurons are statistically identical, and we regard the internal state u_1 of the first neuron as the output of the network. Let us define the correlation coefficient C between the input and output pulse trains (Palm et al., 1988). Firstly, to incorporate the effect of the firing delay d_f of the FN model, which is the time lag of firing since an input pulse is injected, the shift $t \to t - d_f$ is applied to the time series of the output pulse train. Then the time series are divided into n bins of the width Δ, and the number of pulses in the i-th bin is denoted as X_i and Y_i for the input and output pulses, respectively. Note that the width Δ is sufficiently small so that X_i and Y_i take the value 0 or 1. Then $X = \sum X_i$ and $Y = \sum Y_i$ are the numbers of input and output pulses respectively, and $Z = \sum X_i Y_i$ is the number of coincident firings. The correlation coefficient C between the input and output pulse train is defined as

$$C = \frac{Z - (XY)/n}{\sqrt{X(1 - X/n)Y(1 - Y/n)}} \in [-1, 1]. \tag{6}$$

Consider the periodic input with the frequency f such that

$$X_i = \begin{cases} 1 & \text{if} \quad i\Delta \bmod f^{-1} < \Delta \\ 0 & \text{otherwise} \end{cases} . \tag{7}$$

If the output series Y_i is identical with X_i, namely, if the relation $X_i = Y_i$ is satisfied for all i, the correlation coefficient C takes the value 1. If the output series Y_i has no correlation with X_i, the correlation coefficient C takes the value 0 in the large n limit.

3 The case of $d_p = 0$: Stochastic resonance in the coupled system

In this section, we briefly describe the results for $d_p = 0$ (Kanamaru et al., 1999). As previously mentioned, the frequency of the input pulse train is fixed at $f = 0.1$. Firstly, the system with $N = 2$ is examined for simplicity.

The dependence of the correlation coefficient C on the noise intensity D is shown in Fig. 1, for the coupling strength $w = 0$, 0.1, and 0.5. For $w = 0$, the system reduces to a single neuron, and the correlation coefficient C shows the characteristic of Stochastic Resonance, namely, the existence of an optimal noise intensity D_0 which maximizes C. For $w = 0.1$ and 0.5, it is observed that D_0 increases with the increase of w.

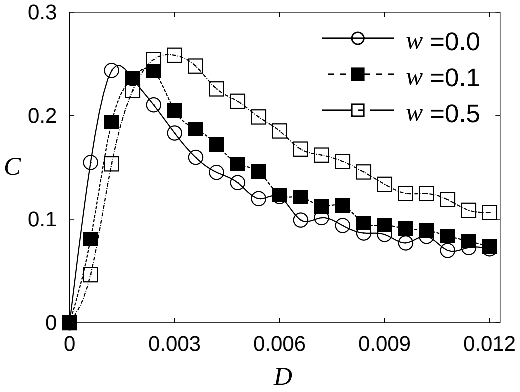

Fig. 1. The dependence of the correlation C on the noise intensity D for the propagational time delay $d_p = 0$.

In Fig. 2, the dependence of D_0 on the coupling strength w is investigated numerically. It is observed that the optimal noise intensity D_0 increases monotonically with the increase of w, and it converges to about 0.0028. In the following, this limit value is denoted by $D_0(\infty)$.

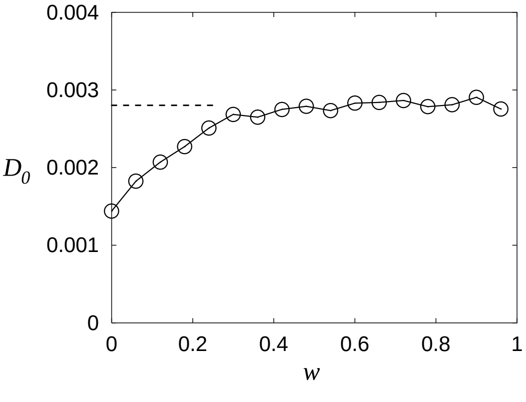

Fig. 2. The dependence of the optimal noise intensity D_0 on the coupling strength w for the propagational time delay $d_p = 0$.

To consider the dependence of $D_0(\infty)$ on the number N of neurons, we introduce $\boldsymbol{x^{(i)}} = (u_i, v_i)^t$, $\boldsymbol{\xi^{(i)}} = (\xi_i, 0)^t$, and a two dimensional diagonal matrix A with diagonal components $A_{11} = 1$ and $A_{22} = 0$, we rewrite the coupled FitzHugh-Nagumo equation as

$$\dot{\boldsymbol{x}}^{(i)} = \boldsymbol{F}(\boldsymbol{x^{(i)}}) + w\boldsymbol{G}_i(t) + \boldsymbol{\xi^{(i)}}, \tag{8}$$

$$\boldsymbol{G}_i(t) = \begin{cases} A\left(\dfrac{1}{N-1}\displaystyle\sum_{j\neq i}\left(\boldsymbol{x^{(j)}} - \boldsymbol{x^{(i)}}\right)\right) & \text{if } N \geq 2 \\ 0 & \text{if } N = 1 \end{cases}, \tag{9}$$

$$\langle \xi_i(t)\xi_j(t')\rangle = D\delta_{ij}\delta(t-t'), \tag{10}$$

$$i, j = 1, 2, \cdots, N,$$

where $\boldsymbol{F}(\boldsymbol{x^{(i)}})$ describes the dynamics of i-th neuron. Let us define the mean value $\boldsymbol{X}$ and the deviation $\delta\boldsymbol{x^{(i)}}$ from $\boldsymbol{X}$ as

$$\boldsymbol{X} = \frac{1}{N}\sum_i \boldsymbol{x^{(i)}}, \tag{11}$$

$$\delta\boldsymbol{x^{(i)}} = \boldsymbol{x^{(i)}} - \boldsymbol{X}, \tag{12}$$

then, for large w, $\boldsymbol{X}$ and $\delta\boldsymbol{x^{(i)}}$ obeys

$$\dot{\boldsymbol{X}} = \boldsymbol{F}(\boldsymbol{X}) + \sum_{i=1}^{N}\frac{\boldsymbol{\xi^{(i)}}}{N} + O(|\delta\boldsymbol{x^{(i)}}|^2), \tag{13}$$

$$\dot{\delta x}_1^{(i)} = -\frac{w}{(1-N^{-1})}\delta x_1^{(i)} + \xi^{(i)} - \sum_{j=1}^{N}\frac{\xi^{(j)}}{N}, \tag{14}$$

$$\dot{\delta x}_2^{(i)} = \delta x_1^{(i)} - b\delta x_2^{(i)}, \tag{15}$$

where b is the parameter of the FN model (2). Thus the variances of $\delta x_1^{(i)}$ and $\delta x_2^{(i)}$ are estimated to be

$$\langle(\delta x_1^{(i)})^2\rangle \simeq \frac{(1-N^{-1})^2}{2w}D, \tag{16}$$

$$\langle(\delta x_2^{(i)})^2\rangle \simeq \frac{\langle(\delta x_1^{(i)})^2\rangle}{b(b+(1-N^{-1})^{-1}w)} \sim \frac{1}{w^2}. \tag{17}$$

Eqs. (12), (13), (16), and (17) indicate that the dynamics of each neuron for large w approaches the dynamics of one neuron with the scaled noise intensity D/N. So, between the optimal noise intensity $D_0^{(N)}(\infty)$ for N neurons and $D_0^{(1)}(\infty)$ for one neuron, the relation

$$D_0^{(N)}(\infty) = ND_0^{(1)}(\infty) \tag{18}$$

holds.

In Fig. 3, the numerically obtained optimal noise intensity $D_0(\infty)$ is plotted against the number N of neurons, where $D_0(\infty)$ is estimated with the coupling strength $w = 1.0$, which is large enough for the saturation of D_0. A good agreement with the analytical result (18) is observed.

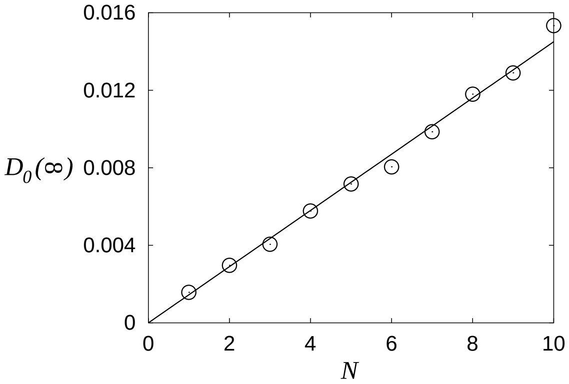

Fig. 3. The dependence of the optimal noise intensity $D_0(\infty)$ on the number N of neurons for the propagational time delay $d_p = 0$.

4 The case of $d_p \neq 0$: Noise-induced deterministic firing

In this section, the system with $d_p \neq 0$ is treated. Firstly, the system with $N = 2$ is considered for simplicity. The frequency of the input pulse train is fixed at $f = 0.1$ again.

For $d_p = 10$, the firing at some moment may affect the firing in the next period of $S(f;t)$ since $1/f = 10$. The dependence of the correlation coefficient C on the noise intensity D for $w = 0$, 0.12 and 0.16 with $d_p = 10$ is shown in Fig. 4. Similar to the case of $d_p = 0$, the correlation coefficient C has a peak at an

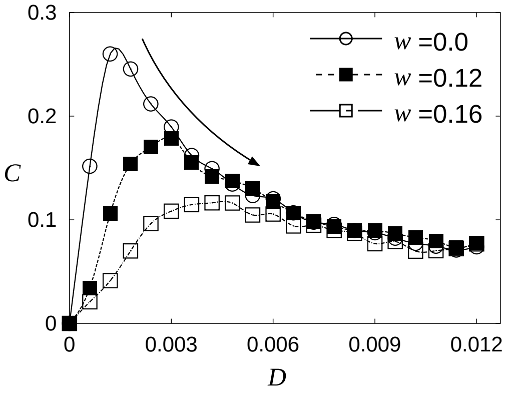

Fig. 4. The dependence of the correlation C on the noise intensity D for the propagational time delay $d_p = 10$.

optimal noise intensity D_0 for $w = 0$, and D_0 increases with the increase of w. But it is also observed that the peak value of C decreases with the increase of w, unlike the case of $d_p = 0$. This difference of the behavior for $d_p = 0$ and 10 may come from the fact that the synchronized solution $u_1(t) = u_2(t - d_p)$ and $u_2(t) = u_1(t - d_p)$ cannot lock to the input pulse train $S(f;t)$ for $d_p = 10$.

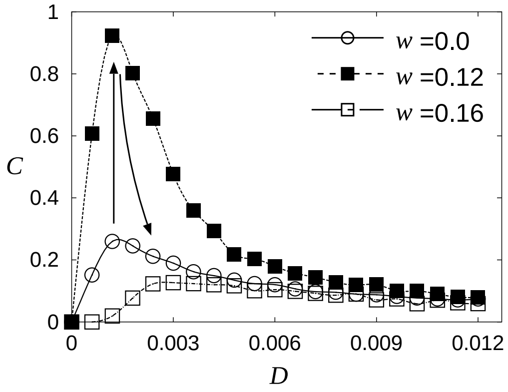

Fig. 5. The dependence of the correlation C on the noise intensity D for the propagational time delay $d_p = 10 - d_f \sim 9.7$.

Next, we treat the case with $d_p = 1/f - d_f \sim 9.7$. Note that in this case an output pulse, which fires with the delay d_f after an input pulse is injected, can synchronize with the next input pulse. The correlation coefficient C for $w = 0$, 0.12, and 0.16 with $d_p = 9.7$ is plotted in Fig. 5, where each C shows the existence of the maximum. Note that the maximum of C for $w = 0.12$ reaches almost 1, which indicates that for $D \sim 0.001$, noise-induced deterministic firing (NIDF), namely, a 1:1 phase locking between the input and the output, takes place. For $w = 0.16$, this locking behavior is broken and the maximum decreases to about 0.1.

To investigate the range of w where NIDF takes place, the correlation coefficient C with the fixed noise intensity $D = 0.001$ for $d_p = 0$, 9.7, and 10 is plotted against w in Fig. 6. It shows that the coupling strength w which enables NIDF is around $w = 0.12$ for $d_p = 9.7$. NIDF also takes place for $N > 2$ with $d_p = 9.7$ and $w \sim 0.12$ (data not shown), but its optimal noise intensity D_0 has a dependence on the number N of neurons. D_0 is shown as a function of N in Fig. 7. The monotonic increase of D_0 with N, and the convergence of D_0 to about 0.002 is observed.

5 Competition of two assemblies under noise

Using the preceding properties of SR in a coupled FN equation with a propagational time delay, we construct a network composed of two assemblies, in

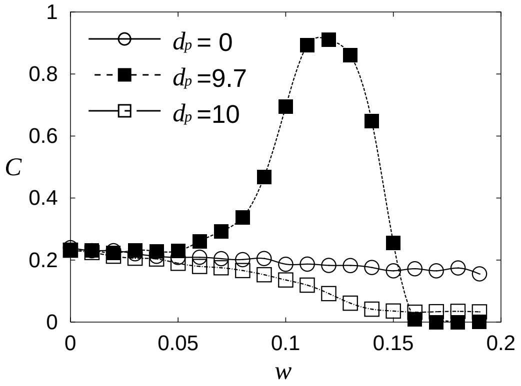

Fig. 6. The dependence of the correlation coefficient C on the coupling strength w for $d_p = 0$, 9.7, and 10 with $D = 0.001$.

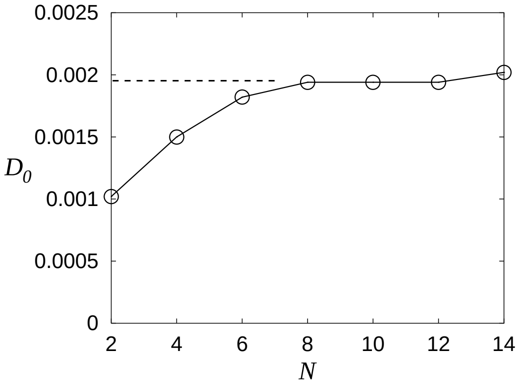

Fig. 7. The dependence of optimal noise intensity D_0 on the number N of neuron for the noise-induced deterministic firing.

which a competitive behavior takes place by controlling the noise intensity. Firstly, let us define the superimposed periodic pulse train (SPPT) as

$$T(t) = \max_{1 \le i \le m} S(f_i; t), \tag{19}$$

where m is the number of periodic components, and f_i is the frequency of each periodic component. In the following, we set $m = 2$, $f_1 = 0.1$, $f_2 = f_1/\sqrt{2}$, and $I = 0.15$. Note that the height I of $T(t)$ is so small that it cannot make each neuron generate a pulse without the noise. The SPPT $T(t)$ is injected to the network composed of two assemblies, shown in Fig. 8. The assembly 1 is composed of two neurons, namely, Eqs. (1) and (2) with $N = N_1 = 2$, $d_p = d_1 \equiv 1/f_1 - d_f$ and $T(t)$ instead of $S(f;t)$, and assembly 2 is composed of eight neurons, namely, Eqs. (1) and (2) with $N = N_2 = 8$, $d_p = d_2 \equiv 1/f_2 - d_f$ and $T(t)$ instead of $S(f;t)$. Note that there is a neuron which belongs to both assemblies simultaneously, and we regard its output as the output of the network. The coupling strength w is set at $w = 0.12$ so that NIDF takes place with a suitable noise intensity. By the definition of the delay d_1 and d_2, it is expected that NIDFs with frequency f_1 and f_2 take place in the assembly 1

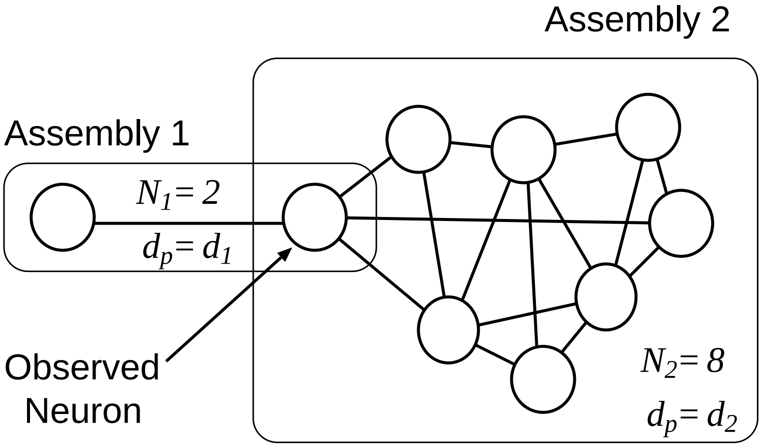

Fig. 8. A network composed of two assemblies.

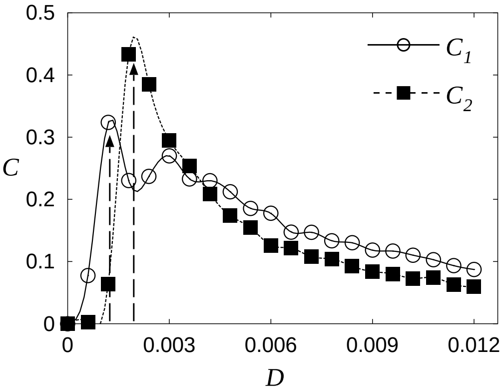

Fig. 9. The dependence of the correlations C_1 and C_2 on the noise intensity D.

and 2, respectively. Then two correlation coefficients C_1 and C_2 are defined as the correlations between the output pulse train of the network and the periodic pulse train $S(f_1;t)$ and $S(f_2;t)$, respectively. A large C_1 indicates that the observed neuron is dominated by the synchronized oscillation in the assembly 1, and a large C_2 indicates the domination of the assembly 2.

The dependence of C_1 and C_2 on the noise intensity D are shown in Fig. 9, which indicates that the optimal noise intensity D_0 is $D_0 \sim 0.001$ for C_1, and $D_0 \sim 0.002$ for C_2. Note that C_1 is suppressed when C_2 is around the maximum value. The difference between optimal noise intensities for C_1 and C_2, is caused by the fact that the optimal noise intensity D_0 depends on the number N of neurons (see Fig. 7).

The above phenomenon shows some new features of noisy pulse neural networks. The dominant frequency of the SPPT $T(t)$ in the network is controlled by the noise intensity. In other words, the SPPT $T(t)$ is separated to each periodic component by controlling the noise intensity. This implies that the fluctuation in a network might be used as a parameter of its dynamics. Secondly, the synchronously oscillating assembly is rearranged by controlling the noise intensity. For example, for $D \sim 0.001$, the observed neuron belongs mainly to the assembly 1 in which each neuron fires synchronously with frequency f_1, and for $D \sim 0.002$, it belongs to the assembly 2 in which the periodic firing with frequency f_2 is dominant. These new features suggest that the fluctuations in the brain might have a functional role in the information processing.

6 Conclusions and discussions

Concerning SR, new features of the noisy pulse neural network with propagational time delay are reported. When the propagational time delay d_p and the frequency f of the periodic input pulse train satisfy the relationship $d_p = 1/f$, an optimal noise intensity which maximizes the correlation coefficient is increased with the increase of the coupling strength, and the peak value of the correlation coefficient decreases with the increase of the coupling strength. For the delay of propagation $d_p = 1/f - d_f$ dependent on the firing delay d_f, a deterministic firing is induced at the optimal noise intensity, which increases with the increase of the number of neurons.

Using these properties, a network composed of two assemblies is constructed. It separates a superimposed periodic pulse train, and its dynamics can be controlled by noise, In this network, the rearrangement of the synchronously oscillating assembly by controlling the noise intensity is observed.

If the environment in the brain where each neuron operates is highly fluctuating due to thermal noise, internal chaos, numerous synaptic inputs, or other possible sources of noise, the above results suggest that such fluctuations might play a similar role as a parameter of dynamical systems, and be significant for the information processing in the brain.

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